

Critical Behavior of Period Doubling in Coupled Area-Preserving Maps

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Abstract

We study the critical behavior of period doublings in N symmetrically coupled area-preserving maps for many-coupled cases with $N > 3$. It is found that the critical scaling behaviors depend on the range of coupling interaction. In the extreme long-range case of global coupling, in which each area-preserving map is coupled to all the other area-preserving maps with equal strength, there exist an infinite number of bifurcation routes in the parameter plane, each of which ends at a critical point. The critical behaviors, which vary depending on the type of bifurcation routes, are the same as those for the previously-studied small N cases ($N = 2, 3$), independently of N . However, for any other non-global coupling cases of shorter range couplings, there remains only one bifurcation route ending at the zero-coupling critical point, at which the N area-preserving maps become uncoupled. The critical behavior at the zero-coupling point is also the same as that for the small N cases ($N = 2, 3$), independently of the coupling range.

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I. INTRODUCTION

Period doubling has been extensively studied in area-preserving maps of McMillan form [1],

$$\begin{aligned}x(t+1) &= -y(t) + f(x(t)), \\ y(t+1) &= x(t),\end{aligned}\tag{1.1}$$

where $(x(t), y(t))$ is a state vector at a discrete time t and f is a nonlinear function. A well-known example is the Hénon's quadratic map with $f(x) = 1 - ax^2$ [2]. As the nonlinearity parameter a increases, an initially stable orbit may lose its stability and give birth to a stable period-doubled orbit. An infinite sequence of such period-doubling bifurcations accumulates at a finite parameter value and exhibits a universal asymptotic behavior. However the asymptotic scaling behaviors for the area-preserving case [3–9] are different from those for the one-dimensional dissipative case [10].

An interesting question is whether the scaling results of area-preserving maps extend to higher-dimensional volume-preserving maps. Thus period doubling in four-dimensional volume-preserving maps has attracted much interest in recent years [9,11–15]. It has been found in Refs. [13–15] that the critical scaling behaviors of period doublings for two symmetrically coupled area-preserving maps are much richer than those for the uncoupled area-preserving case. An infinite number of critical points form a critical Cantor set in the space of the nonlinearity and coupling parameters. The critical behaviors vary depending on the type of bifurcation routes to the critical points. These scaling results hold also for the three-coupled case [16].

It is also interesting to study whether or not the scaling behaviors for the two- and three-coupled cases extend to arbitrary N -coupled ($N = 2, 3, \dots$) cases. Thus we study the critical behaviors of period doublings in many coupled cases with $N > 3$ and compare them with those for the small N cases ($N = 2, 3$). In Sec. II we introduce N symmetrically coupled area-preserving maps and discuss their general properties. Stability of orbits in this N -coupled map is also discussed in Sec. III. We study the critical scaling behaviors of period doublings

in Sec. IV. It is found that the critical behaviors for many-coupled cases with $N > 3$ depend on the coupling range. In the extreme long-range case of global coupling, the stable regions of in-phase orbits of period 2^n ($n = 0, 1, 2, \dots$) form a “bifurcation tree” in the space of the nonlinearity and coupling parameters, like the small N cases ($N = 2, 3$). Consequently there exist an infinite number of bifurcation routes in the parameter plane, each of which ends at a critical point. The critical scaling behaviors, which vary depending on the type of bifurcation routes, are also the same as those for the small N cases ($N = 2, 3$), irrespectively of N . However, for any other non-global coupling cases of shorter range couplings only one bifurcation route ending at the zero-coupling critical point is left in the parameter plane. The critical scaling at the zero-coupling critical point is also the same as that for the small N cases ($N = 2, 3$), irrespectively of the coupling range. Finally, a summary is given in Sec. V.

II. COUPLED AREA-PRESERVING MAPS

In this section we discuss general properties of N coupled area-preserving maps in which the coupling extends to the K th [$1 \leq K \leq \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N] neighbor(s) with equal strength.

Consider N symmetrically coupled area-preserving maps with a periodic boundary condition,

$$T : \begin{cases} \vec{x}(t+1) = -\vec{y}(t) + \vec{F}(\vec{x}(t)), \\ \vec{y}(t+1) = \vec{x}(t), \end{cases} \quad (2.1)$$

where $\vec{x} = (x_1, \dots, x_N)$, $\vec{y} = (y_1, \dots, y_N)$, $\vec{F}(\vec{x}) = (F_1(\vec{x}), \dots, F_N(\vec{x}))$, and N is a positive integer larger than or equal to 2. Here $z_m = (x_m, y_m)$ ($m = 1, \dots, N$) is the state vector of the m th element of the N -coupled map T , and the periodic boundary condition imposes $z_m = z_{m+N}$ for all m . The m th component $F_m(\vec{x})$ of the vector-valued function $\vec{F}(\vec{x})$ is given by

$$\begin{aligned}
F_m(\vec{x}) &= F(x_m, x_{m+1}, \dots, x_{m-1}) \\
&= f(x_m) + g(x_m, x_{m+1}, \dots, x_{m-1}),
\end{aligned} \tag{2.2}$$

where f is the nonlinear function of the uncoupled area-preserving map (1.1), and g is a coupling function obeying a condition

$$g(x, \dots, x) = 0 \text{ for any } x. \tag{2.3}$$

Thus the N -coupled map T becomes:

$$T : \begin{cases} x_m(t+1) = -y_m(t) + f(x_m(t)) \\ \quad \quad \quad + g(x_m(t), \dots, x_{m-1}(t)), \\ y_m(t+1) = x_m(t), \quad m = 1, \dots, N. \end{cases} \tag{2.4}$$

This N -coupled map T is called a symmetric map [13–16] since it has a cyclic permutation symmetry such that

$$\sigma^{-1}T\sigma(\vec{z}) = T(\vec{z}) \text{ for all } \vec{z}, \tag{2.5}$$

where $\vec{z} = (z_1, \dots, z_N)$, σ is a cyclic permutation of \vec{z} such that $\sigma\vec{z} = (z_2, \dots, z_1)$, and σ^{-1} is its inverse. The set of all fixed points of σ is a two-dimensional (2D) subspace of the $2N$ dimensional state space, on which

$$x_1 = \dots = x_N, \quad y_1 = \dots = y_N. \tag{2.6}$$

It follows from Eq. (2.5) that the cyclic permutation σ commutes with the symmetric map T , i.e., $\sigma T = T\sigma$. Hence the 2D subspace becomes invariant under T , i.e., if a point \vec{z} lies on the 2D subspace, then its image $T\vec{z}$ also lies on it. An orbit is called an in-phase orbit if it lies on the 2D invariant subspace, i.e., it satisfies

$$\begin{aligned}
x_1(t) &= \dots = x_N(t) \equiv x^*(t), \\
y_1(t) &= \dots = y_N(t) \equiv y^*(t) \text{ for all } t.
\end{aligned} \tag{2.7}$$

Otherwise it is called an out-of-phase orbit. Here we study only in-phase orbits. They can be easily found from the uncoupled area-preserving map (1.1) since the coupling function g obeys the condition (2.3).

The Jacobian matrix DT of the N -coupled map T is:

$$DT = \begin{pmatrix} D\vec{\mathbf{F}} & -I_N \\ I_N & 0 \end{pmatrix}, \quad (2.8)$$

where $D\vec{\mathbf{F}}$ is the Jacobian matrix of the function $\vec{\mathbf{F}}(\vec{\mathbf{x}})$, I_N is the $N \times N$ identity matrix, and 0 is the $N \times N$ null matrix. Since $\text{Det}(DT) = 1$, the map T is a $2N$ dimensional volume-preserving map. Furthermore, if $D\vec{\mathbf{F}}$ is a symmetric matrix, i.e., $D\vec{\mathbf{F}}^t = D\vec{\mathbf{F}}$ (t denotes transpose), then the map T is a symplectic map because its Jacobian matrix satisfies the relation $DT^t J DT = J$ [17], where

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}. \quad (2.9)$$

The map T is also reversible since it can be factored into the product of two involutions [18], $T = (TS)S$ [involutions satisfy $(TS)^2 = S^2 = I$ (identity map)], where

$$S : \begin{cases} \vec{\mathbf{x}}(t+1) = \vec{\mathbf{x}}(t), \\ \vec{\mathbf{y}}(t+1) = -\vec{\mathbf{y}}(t) + \vec{\mathbf{F}}(\vec{\mathbf{x}}(t)), \end{cases} \quad (2.10)$$

$$TS : \begin{cases} \vec{\mathbf{x}}(t+1) = \vec{\mathbf{y}}(t), \\ \vec{\mathbf{y}}(t+1) = \vec{\mathbf{x}}(t), \end{cases} \quad (2.11)$$

Such a decomposition represents a time-reversal symmetry because the map T is conjugate to its inverse T^{-1} by the involution S such that $STS = T^{-1}$. The sets of the fixed points of the involutions, called the symmetry sets, are:

$$\text{Fix}(S) = \{(\vec{\mathbf{x}}, \vec{\mathbf{y}}) | \vec{\mathbf{y}} = \frac{\vec{\mathbf{F}}(\vec{\mathbf{x}})}{2}\}, \quad \text{Fix}(TS) = \{(\vec{\mathbf{x}}, \vec{\mathbf{y}}) | \vec{\mathbf{y}} = \vec{\mathbf{x}}\}. \quad (2.12)$$

An orbit is called a symmetric orbit if it is invariant under the involution S ; otherwise it is called an asymmetric orbit. It is easy to see that a symmetric periodic orbit has two symmetric points on the symmetry sets [18]: a symmetric orbit of even period has two points on one symmetry set and none on the other, but a symmetric orbit of odd period has one point on each symmetry set. Here we study only symmetric periodic orbits.

Expressing the N -coupled map (2.4) in the form of second-order difference equations, we have:

$$\begin{aligned}
T : x_m(t+1) + x_m(t-1) &= F_m(\vec{x}(t)) \\
&= f(x_m(t)) \\
&\quad + g(x_m(t), \dots, x_{m-1}(t)), \\
m &= 1, \dots, N.
\end{aligned} \tag{2.13}$$

Consider an element, say the m th element, in this N -coupled map T . Then the $(m \pm \delta)$ th elements are called the δ th neighbors of the m th element, where $1 \leq \delta \leq \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N . If coupling extends to the K th neighbor(s), then the number K is called the range of the coupling interaction.

A general form of coupling for odd N ($N \geq 3$) is given by

$$\begin{aligned}
g(x_1, \dots, x_N) &= \frac{c}{2K+1} \sum_{l=-K}^K [u(x_{1+l}) - u(x_1)], \\
&= c \left[\frac{1}{2K+1} \sum_{l=-K}^K u(x_{1+l}) - u(x_1) \right], \\
K &= 1, \dots, \frac{N-1}{2},
\end{aligned} \tag{2.14}$$

where c is a coupling parameter and u is a function of one variable. Here the coupling extends to the K th neighbor(s) with equal coupling strength, and the function g satisfies the condition (2.3) because the value of g for $x_1 = \dots = x_N$ becomes zero. The extreme long-range interaction for $K = \frac{N-1}{2}$ is called a “global” coupling, for which the coupling function g becomes

$$\begin{aligned}
g(x_1, \dots, x_N) &= \frac{c}{N} \sum_{m=1}^N [u(x_m) - u(x_1)] \\
&= c \left[\frac{1}{N} \sum_{m=1}^N u(x_m) - u(x_1) \right].
\end{aligned} \tag{2.15}$$

This is a kind of mean-field coupling, in which each element is coupled to all the other elements with equal coupling strength. All the other couplings with $K < \frac{N-1}{2}$ (e.g., nearest-

neighbor coupling with $K = 1$) will be referred to as non-global couplings. The $K = 1$ case for $N = 3$ corresponds to both the global coupling and the nearest-neighbor coupling.

We next consider the case of even N ($N \geq 2$). The form of coupling of Eq. (2.14) holds for the cases of non-global coupling with $K = 1, \dots, \frac{N-2}{2}$ ($N \geq 4$). The global coupling for $K = \frac{N}{2}$ ($N \geq 2$) also has the form of Eq. (2.15), but it cannot have the form of Eq. (2.14) since there exists only one farthest neighbor for $K = \frac{N}{2}$, unlike the odd N case. The $K = 1$ case for $N = 2$ also corresponds to the nearest-neighbor coupling as well as to the global coupling, like the $N = 3$ case.

III. STABILITY OF IN-PHASE ORBITS

In this section we study stability of in-phase orbits in N coupled area-preserving maps. In case of global coupling the stability region of an in-phase orbit in the parameter plane is the same independently of N , whereas for the other non-global coupling cases of shorter range couplings it depends on the coupling range K .

The stability analysis of an orbit in many coupled maps is conveniently carried out by Fourier-transforming with respect to the discrete space $\{m\}$ [19]. Consider an orbit $\{\vec{x}(t)\} \equiv \{x_m(t) ; m = 1, \dots, N\}$ of the N coupled map (2.13). The discrete spatial Fourier transform of the orbit is:

$$\begin{aligned} \mathcal{F}[x_m(t)] &\equiv \frac{1}{N} \sum_{m=1}^N e^{-2\pi i m j / N} x_m(t) = \xi_j(t), \\ j &= 0, 1, \dots, N-1. \end{aligned} \tag{3.1}$$

The Fourier transform $\xi_j(t)$ satisfies $\xi_j^*(t) = \xi_{N-j}(t)$ ($*$ denotes complex conjugate), and the wavelength of a mode with index j is $\frac{N}{j}$ for $j \leq \frac{N}{2}$ and $\frac{N}{N-j}$ for $j > \frac{N}{2}$.

To determine the stability of an in-phase orbit $[x_1(t) = \dots = x_N(t) \equiv x^*(t)$ for all $t]$, we consider an infinitesimal perturbation $\{\delta x_m(t)\}$ to the in-phase orbit, i.e., $x_m(t) = x^*(t) + \delta x_m(t)$ for $m = 1, \dots, N$. Linearizing the N -coupled map (2.13) at the in-phase orbit, we obtain:

$$\begin{aligned}
\delta x_m(t+1) + \delta x_m(t-1) &= f'(x^*(t)) \delta x_m(t) \\
&+ \sum_{l=1}^N G_l(x^*(t)) \delta x_{l+m-1}(t),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
f'(x) &= \frac{df}{dx}, \quad G_l(x) \equiv \left. \frac{\partial g(\sigma^{(m-1)} \vec{x})}{\partial x_{l+m-1}} \right|_{x_1=\dots=x_N=x} \\
&= \left. \frac{\partial g(\vec{x})}{\partial x_l} \right|_{x_1=\dots=x_N=x}.
\end{aligned} \tag{3.3}$$

Hereafter the functions G_l 's will be called “reduced” coupling functions of $g(\vec{x})$.

Let $\delta \xi_j(t)$ be the Fourier transform of $\delta x_m(t)$, i.e.,

$$\begin{aligned}
\delta \xi_j(t) &= \mathcal{F}[\delta x_m(t)] = \frac{1}{N} \sum_{m=1}^N e^{-2\pi i m j / N} \delta x_m(t), \\
j &= 0, 1, \dots, N-1.
\end{aligned} \tag{3.4}$$

Then the Fourier transform of Eq. (3.2) becomes:

$$\begin{aligned}
\delta \xi_j(t+1) + \delta \xi_j(t-1) &= [f'(x^*(t)) \\
&+ \sum_{l=1}^N G_l(x^*(t)) e^{2\pi i (l-1)j/N}] \\
&\times \delta \xi_j(t), \\
j &= 0, 1, \dots, N-1.
\end{aligned} \tag{3.5}$$

This equation can be also put into the following form:

$$\begin{pmatrix} \delta \xi_j(t+1) \\ \delta \xi_j(t) \end{pmatrix} = L_j(t) \begin{pmatrix} \delta \xi_j(t) \\ \delta \xi_j(t-1) \end{pmatrix}, \quad j = 0, 1, \dots, N-1, \tag{3.6}$$

where

$$L_j(t) = \begin{pmatrix} f'(x^*(t)) + \sum_{l=1}^N G_l(x^*(t)) e^{2\pi i (l-1)j/N} & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.7}$$

Note that the determinant of L_j is one, i.e., $\text{Det}(L_j) = 1$.

Stability of an in-phase orbit of period q is determined by iterating Eq. (3.6) q times:

$$\begin{pmatrix} \delta\xi_j(t+q) \\ \delta\xi_j(t+q-1) \end{pmatrix} = M_j \begin{pmatrix} \delta\xi_j(t) \\ \delta\xi_j(t-1) \end{pmatrix},$$

$$j = 0, 1, \dots, N-1, \quad (3.8)$$

where

$$M_j = \prod_{k=t}^{t+q-1} L_j(k). \quad (3.9)$$

That is, the stability of each mode with index j is determined by the 2×2 matrix M_j . Since $\text{Det}(M_j) = 1$, each matrix M_j has a reciprocal pair of eigenvalues, λ_j and λ_j^{-1} . These eigenvalues are called the stability multipliers of the mode with index j . We also associate with a pair of multipliers $(\lambda_j, \lambda_j^{-1})$ a stability index,

$$\rho_j = \lambda_j + \lambda_j^{-1}, \quad j = 0, 1, \dots, N-1, \quad (3.10)$$

which is just the trace of M_j , i.e., $\rho_j = \text{Tr}(M_j)$.

It follows from the condition (2.3) that the reduced coupling functions satisfy

$$\sum_{l=1}^N G_l(x) = 0. \quad (3.11)$$

Hence the matrix of Eq. (3.7) for $j = 0$ becomes:

$$L_0(t) = \begin{pmatrix} f'(x^*(t)) & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.12)$$

This is just the Jacobian matrix of the uncoupled area-preserving map. Therefore the stability index ρ_0 of the $j = 0$ mode is just that for the case of the area-preserving map, i.e., ρ_0 depends only on the nonlinearity parameter a . While there is no coupling effect on ρ_0 , coupling generally affects the other stability indices ρ_j 's ($j \neq 0$).

In case of the global coupling of Eq. (2.15), the reduced coupling functions become:

$$G_l(x) = \begin{cases} (1-N)G(x) & \text{for } l = 1, \\ G(x) & \text{for } l \neq 1, \end{cases} \quad (3.13)$$

where $G(x) = \frac{c}{N}u'(x)$. Substituting G_l 's into the $(1, 1)$ entry of the matrix $L_j(t)$, we have:

$$\sum_{l=1}^N G_l(x) e^{2\pi i(l-1)j/N} = \begin{cases} 0 & \text{for } j = 0, \\ -c u'(x) & \text{for } j \neq 0. \end{cases} \quad (3.14)$$

Hence all stability indices ρ_j 's of modes with non-zero index j ($j \neq 0$) become real and the same, i.e., $\rho_1 = \dots = \rho_{N-1}$. Thus there exist only two independent real stability indices ρ_0 and ρ_1 , the values of which are also independent of N .

We next consider the non-global coupling of the form (2.14) and define

$$G(x) \equiv \frac{c}{2K+1}u'(x), \quad (3.15)$$

where $1 \leq K \leq \frac{N-2}{2}$ ($\frac{N-3}{2}$) for even (odd) N larger than 3. Then we have

$$G_l(x) = \begin{cases} -2KG(x) & \text{for } l = 1, \\ G(x) & \text{for } 2 \leq l \leq 1+K \text{ or} \\ & \text{for } N+1-K \leq l \leq N, \\ 0 & \text{otherwise.} \end{cases} \quad (3.16)$$

Substituting the reduced coupling functions into the matrix $L_j(t)$ of Eq. (3.7), the second term of the $(1, 1)$ entry of $L_j(t)$ becomes:

$$\sum_{l=1}^N G_l(x) e^{2\pi i(l-1)j/N} = -S_N(K, j)c u'(x), \quad (3.17)$$

where

$$S_N(K, j) \equiv \frac{4}{2K+1} \sum_{k=1}^K \sin^2 \frac{\pi j k}{N} = 1 - \frac{\sin(2K+1)\frac{\pi j}{N}}{(2K+1)\sin\frac{\pi j}{N}}. \quad (3.18)$$

Hence all stability indices ρ_j 's with non-zero index ($j \neq 0$) become real, but they vary depending on the coupling range K as well as on the mode number j . Since $S_N(K, j) = S_N(K, N-j)$, the real stability indices satisfy

$$\rho_j = \rho_{N-j}, \quad j = 0, 1, \dots, N-1. \quad (3.19)$$

Thus it is sufficient to consider only the case of $0 \leq j \leq \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N . Comparing the expression in Eq. (3.17) with that in Eq. (3.14) for $j \neq 0$, one can easily see that they are the same except for the factor $S_N(K, j)$. Consequently, making a change of the coupling parameter $c \rightarrow \frac{c}{S_N(K, j)}$, the stability index ρ_j for the non-global coupling case of range K becomes the same as that for the global-coupling case.

It follows from the reality of ρ_j that the reciprocal pair of eigenvalues of M_j lies either on the unit circle, or on the real line in the complex plane, i.e., they are a complex conjugate pair on the unit circle, or a reciprocal pair of reals. Each mode with index j is stable if and only if the magnitude of its stability index ρ_j is less than or equal to two ($|\rho_j| \leq 2$), i.e., its stability multipliers are a pair of complex conjugate numbers of modulus unity. A period-doubling (tangent) bifurcation occurs when the stability index ρ_j decreases (increases) through -2 (2), i.e., two eigenvalues coalesce at $\lambda_j = -1$ (1) and split along the negative (positive) real axis.

When the stability index ρ_0 for an in-phase orbit in an N -coupled map decreases through -2 , the in-phase orbit loses its stability via in-phase period-doubling bifurcation, giving rise to the birth of the period-doubled in-phase orbit. Here we are interested in such in-phase period-doubling bifurcations. Thus, for each mode with non-zero index j ($j \neq 0$) we consider a region in the space of the nonlinearity and coupling parameters, in which both modes with indices 0 and j are stable. This stable region is bounded by four bifurcation lines of both modes (i.e., those curves determined by the equations $\rho_0 = \pm 2$ and $\rho_j = \pm 2$), and it will be denoted by U_N .

In case of global coupling, those stable regions coincide, independently of N and j , because all stability indices ρ_j 's of modes with non-zero j are the same, irrespectively of N ; the stable region for this case will be denoted by U_G . An in-phase orbit is stable only when all its modes are stable. For the global-coupling case, U_G itself is the stability region of the in-phase orbit irrespectively of N , because all modes are stable in the region U_G .

However the stable regions U_N 's vary depending on the coupling range K and the mode number j for the non-global coupling cases, i.e., $U_N = U_N(K, j)$. To find the stability region of an in-phase orbit in an N -coupled map with a given K , one may start with the stability region U_G for the global-coupling case. Rescaling the coupling parameter c by a scaling factor $\frac{1}{S_N(K, j)}$ for each non-zero j ($j \neq 0$), the stable region U_G is transformed into a stable region $U_N(K, j)$. Then the stability region of the in-phase orbit is given by the intersection of all such stable regions U_N 's.

IV. CRITICAL BEHAVIOR OF PERIOD DOUBLINGS

We are concerned about the critical scaling behavior of period doublings of in-phase orbits in N symmetrically coupled area-preserving maps (2.13). Small N cases ($N = 2, 3$) have been studied in Refs. [12–16]. It is interesting to study whether or not the critical scalings for the small N cases extend to the large N cases. Thus we study the critical behaviors in the large N cases and find that those for $N > 3$ depend on the coupling range K . For the N -coupled map (2.13), we choose $f(x) = 1 - ax^2$ as the nonlinear function of the uncoupled area-preserving map and consider separately two kinds of couplings, the global and non-global coupling cases.

A. Global Coupling

We first study an N -coupled map with global coupling, in which each element is coupled to all the other elements with equal strength. As shown in Sec. III, all stability indices ρ_j 's of modes with non-zero j are not only the same (i.e., $\rho_1 = \dots = \rho_{N-1}$), but also independent of N for the global-coupling case. Thus the stability diagram of in-phase orbits in the space of the nonlinearity and coupling parameters becomes the same as that for the two-coupled case, independently of N . That is, the stable regions of in-phase orbits of period 2^n ($n = 0, 1, 2, \dots$) form a “bifurcation tree” in the parameter plane. Consequently there exist an infinite number of bifurcation routes in the parameter plane, each of which ends at

a critical point. The critical behaviors, which vary depending on the type of the bifurcation routes, are also the same as those for the two-coupled case, irrespectively of N (for details of the $N = 2$ case, refer to Refs. [13–15]).

As an example, we consider a linearly-coupled case in which the coupling function (2.15) is

$$g(x_1, \dots, x_N) = c \left[\frac{1}{N} \sum_{m=1}^N x_m - x_1 \right]. \quad (4.1)$$

Figure 1 shows the stability diagram of in-phase orbits with period $q = 1, 2, 4$. These low-period orbits can be analytically obtained from the uncoupled second-order difference equation, $x(t+1) + x(t-1) = f(x(t))$ [6].

The period-1 fixed point is

$$x^*(0) = \frac{1}{a}(-1 + \sqrt{1+a}), \quad (4.2)$$

and the independent stability indices are

$$\rho_0(a) = 2(1 - \sqrt{1+a}), \quad (4.3)$$

$$\rho_1(a, c) = \rho_0(a) - c. \quad (4.4)$$

Its stable region in the parameter plane is limited by four bifurcation lines associated with tangent and period-doubling bifurcations of both modes (i.e., those curves determined by the equations $\rho_j = \pm 2$ for $j = 0, 1$).

When the stability index ρ_0 of the fixed point decreases through -2 , it loses its stability via in-phase period-doubling bifurcations, and gives rise to the birth of an in-phase orbit of period 2,

$$x^*(0) = \frac{1}{a}(1 - \sqrt{a-3}), \quad x^*(1) = \frac{1}{a}(1 + \sqrt{a-3}). \quad (4.5)$$

Two independent stability indices of the period-2 orbit are

$$\rho_0(a) = -4a + 14, \quad (4.6)$$

$$\rho_1(a, c) = \rho_0(a) + c(c+4). \quad (4.7)$$

Note that two period-2 stable branches bifurcate out of the period-1 stable region in the parameter plane (see Fig. 1).

An in-phase orbit of period 4,

$$\begin{aligned} x^*(0) &= \left(\frac{\sqrt{a}-2}{a\sqrt{a}} \right)^{\frac{1}{2}}, \quad x^*(1) = \frac{1}{\sqrt{a}}, \\ x^*(2) &= -x^*(0), \quad x^*(3) = x^*(1), \end{aligned} \quad (4.8)$$

bifurcates from the in-phase period-2 orbit at $a = 4$. Its two independent stability indices are

$$\rho_0(a) = 16a(2\sqrt{a} - a) + 2, \quad (4.9)$$

$$\begin{aligned} \rho_1(a, c) &= \rho_0(a) + c^4 + 4\sqrt{a}c^3 + 4(2\sqrt{a} - 1)c^2 \\ &\quad - 8\sqrt{a}(2a - 4\sqrt{a} + 1)c. \end{aligned} \quad (4.10)$$

As shown in Fig. 1, each of the two period-2 stable branches also bifurcates into two period-4 branches. Thus the stability region for this case consists of 4 stable branches.

However it is difficult to obtain analytically higher periodic orbits with period larger than four. Thus one usually resorts to numerical computation. Successive bifurcations of stability regions have been also observed for the case of higher periods [13,15]. That is, each “mother” stable branch bifurcates into two “daughter” stable branches successively in the parameter plane; hereafter we call the direction of the left (right) branch of the two daughter branches L (R) direction. Consequently the stability region of the in-phase orbit of period 2^n ($n = 0, 1, 2, \dots$) consists of 2^n branches. Each branch can be represented by its address $[a_0, \dots, a_n]$, which is a sequence of symbols L and R such that $a_0 = L$ and $a_i = L$ or R for $i \geq 1$.

An infinite sequence of connected stable branches (with increasing period) is called a bifurcation “route” [13,15]. Each bifurcation route is also represented by an infinite sequence of symbols L and R . Hence there exist an infinite number of bifurcation routes in the parameter plane. A bifurcation “path” in a bifurcation route is formed by following a sequence of parameters (a_n, c_n) , at which the in-phase orbit of level n (period 2^n) has some given

stability indices (ρ_0, ρ_1) [13,15]. All bifurcation paths within a bifurcation route converge to an accumulation point (a^*, c^*) . Here the value of a^* is always the same as that of the accumulation point for the area-preserving case (i.e., $a^* = 4.136\,166\,803\,904\dots$), whereas the value of c^* varies depending on the bifurcation routes. Thus each bifurcation route ends at a critical point (a^*, c^*) in the parameter plane.

The nonlinearity-parameter values a_n geometrically converge to a limit value a^* in the limit of large n , i.e.,

$$a_n - a^* \sim \delta_1^{-n} \quad \text{for large } n, \quad (4.11)$$

where the value of δ_1 is always the same as that of the scaling factor δ ($= 8.721\dots$) for the uncoupled area-preserving case, independently of bifurcation routes. However the scaling behaviors associated with the coupling parameter depend on the type of bifurcation routes.

Consider p ($p = 1, 2, \dots$) subsequences $\{c_{pm+j}; m = 0, 1, 2, \dots\}$ ($j = 0, \dots, p-1$) of the coupling-parameter sequence $\{c_n\}$. Then a bifurcation route is called a “period- p ” route [15] if the p subsequences geometrically converge to a limit value c^* in the limit of large m , i.e.,

$$c_{pm+j} - c^* \sim \delta_2^{-m} \quad \text{for large } m, \quad (4.12)$$

where the value of δ_2 is independent of j , but it depends on the type of bifurcation routes.

As an example, consider the case of the lowest period-1 bifurcation routes [13–15]. There exist three kinds of period-1 routes, called the S , A , and E routes. An S route is followed if one goes asymptotically in the same direction (L or R direction). Hence its address has the form, $[u, (L,)^\infty]$ ($\equiv [u, L, L, \dots]$) or $[v, (R,)^\infty]$, where u and v are arbitrary finite sequences of L and R . The coupling-parameter scaling factor for this case is $\delta_2 = 4.000$. Unlike the case of S routes, the direction of an A route asymptotically alternates between the L direction and the R direction. Hence its address has the form, $[w, (L, R,)^\infty]$, where w is an arbitrary finite sequence of L and R . The value of δ_2 is -2.000 for all A routes except for the case of one special route, called the E route. The address of the E route is $[(L, R,)^\infty]$ (i.e., the sequence w for this case is empty). (Hereafter only the routes with non-empty w sequences

will be called A routes.) The limit value of the coupling parameter for the E route is $c^* = 0$, and the scaling factor is $\delta_2 = -4.404$.

In order to see the periodicity of a bifurcation route, a “route sequence” has been introduced in Ref. [15]. It is uniquely determined by the address of the route as follows. If the n th element of the address is the same as the next $(n + 1)$ th element, we assign a 0 to the n th element of the route sequence; otherwise we assign a 1. Thus the route sequence becomes an infinite sequence of two symbols 0 and 1.

As an example we examine the asymptotic patterns of route sequences for the case of the period-1 routes. The route sequence of an S route is $[u', (0,)^\infty]$ because one goes asymptotically in the same direction in an S route; here u' is an arbitrary finite sequence of 0 and 1. On the other hand, the route sequence of an A or the E route is $[w', (1,)^\infty]$ because the direction of an A or the E route asymptotically alternates between the L and R directions. Here w' is an arbitrary non-empty sequence of 0 and 1 for the case of A routes, whereas it is empty for the E route. Note that the route sequence of any period-1 route exhibits eventually a period-1 pattern.

From the asymptotic period- p patterns of the route sequences for the low-period ($p = 1, 2$) routes, it has been conjectured in [15] that if the route sequence of a bifurcation route exhibits an asymptotic period- p ($p = 1, 2, \dots$) pattern, the route becomes a period- p one; otherwise it becomes a random one in which the coupling-parameter values c_n randomly converge to a limit value c^* without any periodicity.

B. Non-global Coupling

Here we study the non-global coupling cases with $K < \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N larger than three. Of the infinite kinds of bifurcation routes for the global coupling case, only the E route ending at the zero-coupling critical point $(a^*, 0)$ is left as a bifurcation route in the parameter plane for all the non-global coupling cases. The critical scaling behaviors near the zero-coupling critical point are also the same as those for the $N = 2$ and 3 cases of global

coupling.

As an example we consider a linearly-coupled, nearest-neighbor coupling case with $K = 1$, in which the coupling function is

$$g(x_1, \dots, x_N) = \begin{cases} \frac{c}{2} (x_2 - x_1) & \text{for } N = 2, \\ \frac{c}{3} (x_2 + x_N - 2x_1) & \text{for } N \geq 3. \end{cases} \quad (4.13)$$

Note that the two- and three-coupled maps for $N = 2, 3$ correspond to the global coupling case. The reduced coupling functions G_l 's ($l = 1, \dots, N$) of g defined in Eq. (3.3) are given by

$$G_2(x) = \frac{c}{2} \equiv G(x), \quad G_1(x) = -G(x) \quad \text{for } N = 2, \quad (4.14)$$

$$\begin{aligned} G_2(x) &= G_N(x) = \frac{c}{3} \equiv G(x), \\ G_1(x) &= -2G(x), \quad G_l(x) = 0 \quad (l \neq 1, 2, N) \quad \text{for } N \geq 3. \end{aligned} \quad (4.15)$$

Substituting G_l 's into the (1,1) entry of the matrix $L_j(t)$ of Eq. (3.7), we obtain:

$$\sum_{l=1}^N G_l(x^*(t)) e^{2\pi i(l-1)j/N} = \begin{cases} -c \delta_{j1} & \text{for } N = 2, 3, \\ -S_N(1, j)c & \text{for } N > 3, \end{cases} \quad (4.16)$$

where δ_{j1} is the Kronecker delta (i.e., δ_{j1} is 0 for $j = 0$ and 1 for $j = 1$), and $S_N(1, j) = \frac{4}{3} \sin^2 \frac{\pi j}{N}$. Making a change of the coupling parameter $c \rightarrow \frac{c}{S_N(1, j)}$ for each non-zero j , the stability index ρ_j for $N > 3$ becomes the same as that for the $N = 2$ and 3 cases of global coupling.

For each mode with non-zero index j , consider the stability region $U_N(1, j)$, in which both modes with indices 0 and j are stable. The stability region U_G for the global-coupling case is independent of both N and j because all stability indices ρ_j 's of modes with non-zero j are the same, independently of N . Rescaling the coupling parameter c with the scaling factor $\frac{1}{S_N(1, j)}$, the stability region U_G for the $N = 2$ and 3 cases of global coupling is transformed

into the stability region $U_N(1, j)$ for $N > 3$. Since the factor S_N is dependent on the index j , the stability region $U_N(1, j)$ varies depending on the index j . Then the stability region of an in-phase orbit, in which all its modes are stable, is given by the intersection of all such stable regions U_N 's.

As an example we consider the case $N = 4$. Figure 2 shows the stability regions of the 2^n -periodic ($n = 1, 2, 3, 4$) orbits for this case. We first note that the scaling factor $\frac{1}{S_4(1, j)}$ has its minimum value $\frac{3}{4}$ at $j = 2$. However $U_4(1, 2)$ itself cannot be the stability region of the in-phase orbit of level 1 (i.e., $n = 1$), because bifurcation curves of different modes with non-zero indices intersect one another. As shown in Fig. 2(a), the branch including a $c = 0$ line segment remains unchanged, whereas the other branch becomes flattened by the bifurcation curve of the mode with $j = 1$. Due to the successive flattening with increasing level n [see Figs. 2(a) and (b)], of the infinite kinds of bifurcation routes for the global-coupling case remains only the E route ending at the zero-coupling point $(a^*, 0)$. Thus only the zero-coupling point is left as a critical point in the parameter plane. Note also that Figs. 2(a) and (b) nearly coincide near the zero-coupling point except for small numerical differences.

We now examine the scaling behaviors near the zero-coupling critical point for the $N = 4$ case of nearest-neighbor coupling. Consider a self-similar sequence of parameters (a_n, c_n) , at which the in-phase orbit of level n has some given stability indices, in the E route for the global-coupling case. Rescaling the coupling parameter with the factor $\frac{3}{4}$, this sequence is transformed into a self-similar one for the $N = 4$ case of nearest-neighbor coupling. Thus the “width” of each stability region in the E route for the case of the global coupling is reduced to that for the $N = 4$ case of nearest-neighbor coupling by the scaling factor $\frac{3}{4}$, whereas the “heights” of all stable regions in the E route remain unchanged [20]. It is therefore obvious that the scaling behaviors near the zero-coupling critical point for the nearest-neighbor coupling case are the same as those for the global coupling case. That is, the height and width h_n and w_n of the stability region of level n geometrically contract in the limit of large n ,

$$h_n \sim \delta_1^{-n}, \quad w_n \sim \delta_2^{-n} \quad \text{for large } n, \quad (4.17)$$

where $\delta_1 = 8.721 \dots$ and $\delta_2 = -4.404$.

The scaling results for the nearest-neighbor coupling case with $K = 1$ extend to all the other non-global coupling cases with $1 < K < \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N . For each non-global coupling case with $K > 1$, consider a mode with index j_{\max} for which the factor $S_N(K, j)$ of Eq. (3.18) becomes the largest one and its stability region $U_N(K, j_{\max})$ including a $c = 0$ line segment. Here the value of j_{\max} varies depending on the range K . Like the nearest-neighbor coupling case with $K = 1$, of the two stable branches the one including the $c = 0$ line segment remains unchanged, whereas the other one becomes flattened by the bifurcation curves of the other modes with non-zero indices. Thus the overall shape of the stability diagram in the parameter plane becomes essentially the same as that for the case with $K = 1$. Consequently only the E -route ending at the zero-coupling point is left as a bifurcation route, and the scaling behaviors near the zero-coupling critical point are also the same as those for the global-coupling case.

V. SUMMARY

The critical behaviors of period doublings in N symmetrically coupled area-preserving maps are studied for many-coupled cases with $N > 3$. It is found that the critical scaling behaviors depend on the coupling range K . For the global-coupling case [$K = \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N], the stable regions of in-phase orbits with period $q = 2^n$ ($n = 0, 1, 2, \dots$) form a “bifurcation tree” in the space of the nonlinearity and coupling parameters, like the small N cases ($N = 2, 3$). Hence there exist an infinite number of bifurcation routes in the parameter plane, each of which ends at a critical point. The critical behaviors, which vary depending on the type of the bifurcation routes, are also the same as those for the small N cases ($N = 2, 3$), independently of N . However, for any other non-global coupling cases [$1 \leq K < \frac{N}{2}$ ($\frac{N-1}{2}$) for even (odd) N], only the E -route ending at the zero-coupling point is left as a bifurcation route in the parameter plane. The critical behaviors at the zero-coupling

critical point are also the same as those for the small N cases ($N = 2, 3$), independently of K .

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- [20] We define the “width” w_n of the stable region of level n in the E route as the length of the boundary line segment associated with the period-doubling bifurcation of the $j = 0$ mode. The “height” h_n is also defined as the length of the $c = 0$ line segment inside the stable region.

FIGURES

FIG. 1. Stability diagram of in-phase orbits in N linearly coupled maps with the global coupling. The horizontal (non-horizontal) solid and dashed lines denote the period-doubling and tangent bifurcation lines of the $j = 0$ (1) mode, respectively. Note that all stability indices ρ_j 's of modes with non-zero j are the same (i.e., $\rho_1 = \dots = \rho_{N-1}$) for the global-coupling case. For other details see the text.

FIG. 2. Stable regions of the in-phase period- 2^n orbits of level $n = 1, 2, 3, 4$ in four linearly coupled maps with the nearest neighbor coupling. The case of $n = 1$ and 2 (3 and 4) is shown in (a) [(b)]. Each stable region is bounded by its solid boundary curves. The period-doubling (tangent) bifurcation curve of the j mode of an in-phase orbit with period q is denoted by a symbol $q_j^{p(t)}$. The values of the scaling factors used in (b) are $\delta_1 = 8.721$ and $\delta_2 = -4.404$.